

Electron Propagation in the Field of Colliding Nuclei at Ultrarelativistic Energies

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Abstract

We calculate the asymptotic high-energy amplitude for electrons scattering at one ion as well as at two colliding ions, respectively, by means of perturbation theory. We show that the interaction with one ion *eikonalizes* and that the interaction with two ions *causally decouples*. We are able to put previous results on perturbative grounds and propose further applications for the obtained rules for interactions on the light cone. The formalism will be of use for the calculation of Coulomb corrections to electron-positron pair creation in heavy ion collisions. Finally we discuss the results and inherent dangers of the employed approximations.

1 Introduction

At ultrarelativistic energies, the theoretical treatment of scattering processes is extremely facilitated. On the one hand, the relevant equations themselves simplify, when terms of order $\mathcal{O}(1/\gamma^2)$ become negligible, on the other hand, the interactions simplify due to causality. In that way, high energy scattering becomes analytically accessible.

Eikonal approximations or optical models usually are formulated for the scattering of a highly energetic particle at a slow or even static center [1][2]. We present a simple transformation of the covariant derivatives that is used to easily solve the opposite case. The transformation of the equations of motion for particles scattered by fast moving charge centers immediately generates the scattered wave describing the particle. Our results coincide with previous calculations performed in this reference frame [3][4].

The summation of ladder graphs is shown to *eikonalize* as well [5]. This was elegantly derived within the method of kinematically decoupling the components of the scattering process, and Lorentz transforming into the respective rest frames [6] which inherently contains the advantages of a fast external potential.

Following a different approach we will exploit the same advantages. We perform a perturbative approach and directly approximate the external potential by its asymptotic

high-energy limit which amounts to saying, that the longitudinal components of the exchanged photons can be discarded.

In doing so, one can directly rederive the amplitude for the scattering at one center and even put the recent result of Segev and Wells [7] for the scattering amplitude for an electron moving in the field of two ultrarelativistic colliding ions on perturbative grounds. Moreover, one is allowed to go beyond their calculations and is provided deeper insight.

The derivations in this paper are formulated for electron scattering, but they can be immediately extended to cover the physically more relevant process of electron-positron pair production. The search for exact analytic expressions describing electron-positron pair production in heavy-ion collisions is motivated by the question whether Coulomb effects only play an inferior role at high energies. Such a conclusion might be drawn from a comparison between second-order perturbation theory results [8] and calculations employing Furry-Sommerfeld-Maue wave functions [9]. It should be mentioned, however, that the Coulomb distortions considered in these calculations only account for one ion, whereas the second ion enters as a perturbation.

2 Scattering of an electron off a fast moving source

2.1 Transformation of the Dirac equation

We are searching for the asymptotic scattering solution of a Dirac particle from a fast moving Coulomb potential in the limit of very large collision energy. In the Lorentz gauge the Liénard-Wiechert potentials for a point charge moving with uniform velocity β in $+z$ direction read

$$A_0 = -\frac{Z\alpha\gamma}{\sqrt{\gamma^2(z-\beta t)^2 + \vec{x}_\perp^2}} \quad (1)$$

$$A_3 = \beta A_0 \quad (2)$$

The equation of motion for the scattered particle becomes

$$\left[\hat{\gamma}_0(i\partial_t - A_0) + \hat{\gamma}_3(i\partial_z + A_3) + \hat{\vec{\gamma}}_\perp \cdot i\vec{\nabla}_\perp - m \right] \psi = 0 \quad (3)$$

We set $c = \hbar = 1$. The charge e of the electron was absorbed into the definition of the potential. We make use of the external field approximation, i.e. we assume that the source is not influenced by the scattered particle and moves on a straight line. This treatment will be justified if the mass of the source particle is very large. To simplify the Dirac equation (3) we use the operator identity [10]

$$(i\partial_x \mp i\partial_x \ln \phi)^n = \phi^{\pm 1} (i\partial_x)^n \phi^{\mp 1} \quad (4)$$

to rewrite the covariant derivatives. We must introduce two fields ϕ' and ϕ for the space and the time component of the vector-potential A^μ

$$\begin{aligned} A_0 &= i\partial_t \ln \phi \\ A_3 &= i\partial_z \ln \phi' \end{aligned} \quad (5)$$

The field ϕ is determined to be

$$\phi = e^{-i \int_{-\infty}^t dt' A_0} \quad (6)$$

The thus transformed Dirac equation reads (see Appendix A)

$$\left[\hat{\gamma}_0 i \partial_t + \hat{\gamma}_3 (i \partial_z - \frac{1}{\beta^2 \gamma^2} A_3) + \hat{\vec{\gamma}}_{\perp} \cdot (i \vec{\nabla}_{\perp} + i \text{grad}_{\perp} \ln \phi) - m \right] \tilde{\psi} = 0 \quad (7)$$

where $\tilde{\psi} = \psi/\phi = e^{i \int_{-\infty}^t dt' A_0} \psi$. The operator identity, together with the field ϕ defined in (6) has led to the elimination of the scalar part of the vector potential, i.e. to the temporal gauge, $A'_0 = 0$. For very large γ one nearly has a purely transverse vector potential $\vec{A}'_{\perp} = i \text{grad}_{\perp} \ln \phi$ which is the negative time integral of the transverse electric field. From classical electrodynamics one knows, that the time integral of the transverse electric field is given by

$$\int_{-\infty}^{\infty} \vec{E}_{\perp} = -2Z\alpha \vec{x}_{\perp} / (\beta x_{\perp}^2) \quad (8)$$

This implies that

$$\int_{-\infty}^{\infty} dt' A_0 = +\frac{1}{\beta} Z\alpha \ln x_{\perp}^2 + \mathcal{C} \quad (9)$$

which reproduces (8) if the transverse gradient operator is applied. \mathcal{C} is an infinite quantity which expresses the divergence of the phases in Coulomb scattering. Furthermore using (6) and (1) it is easy to show, that the transverse vector potential exhibits a Heaviside step function dependence $\sim \theta(t - z) \vec{x}_{\perp} / x_{\perp}^2$ in the limit of very large γ . Now, since t and γ enter symmetrically in the integral, the limit $\gamma \rightarrow \infty$ corresponds to sending the upper bound of the integral to infinity. Therefore, all of the above is applicable and we find

$$\lim_{\gamma \rightarrow \infty} A_0 = +\delta(z - t) Z\alpha \ln x_{\perp}^2 + \mathcal{C}' \quad (10)$$

The Coulomb phase \mathcal{C}' in general will depend on z and t . It can be removed by a gauge transformation, as is easily seen

$$\tilde{\psi}' = e^{-i \int_{-\infty}^t dt' \mathcal{C}'} \tilde{\psi} = e^{+i Z\alpha \theta(t-z) \ln x_{\perp}^2} \tilde{\psi} \quad (11)$$

This gauge transformation was first applied in [11]. The removal of the Coulomb phase yields a short range potential allowing for asymptotic plane wave solutions (see Appendix B).

For $t \neq z$ the t and z dependence in both the transverse vector potential and the transformed spinor $\tilde{\psi}$ vanish in the limit $\gamma \rightarrow \infty$. By inverse transformation we find that ψ solves a free Dirac equation on either side of the light front $t = z$ and can only differ by a phase.

The transformed wave function $\tilde{\psi}$ has the advantage of being continuous on the surface defined by $t = z$. In contrast, the wave function ψ exhibits a discontinuous behaviour at the light front. There is a jump in that component of ψ which couples to $\hat{\gamma}_{-} = \hat{\gamma}_0 - \hat{\gamma}_3$, the

matrix structure of the interaction in the limit $\gamma \rightarrow \infty$. Using this property one directly finds for $\hat{\gamma}_-\psi$ at the discontinuity

$$\hat{\gamma}_-\psi(t-z=0^+) = e^{-iZ\alpha \ln x_\perp^2} \hat{\gamma}_-\psi(t-z=0^-) \quad (12)$$

where we ignored the irrelevant quantity \mathcal{C} .¹ The complement $\hat{\gamma}_+\psi$ of these spinor components, where $\hat{\gamma}_+ = 2\hat{\gamma}_0 - \hat{\gamma}_- = \hat{\gamma}_0 + \hat{\gamma}_+$ is continuous at $t = z$. Both parts of the spinor are coupled via the free Dirac equation on either side of the discontinuity.

By application of the LSZ-reduction formula one finds in general, that at very large scattering energies the S matrix is determined by ϕ , in which we recognize the well known eikonal form [12]. Because of the identity (4) this result holds independently of the power of the momentum entering in the respective wave equation. For that reason the expressions for the S matrices for e.g. spinor or scalar particles only differ by an overall factor.

We first consider the properties of the previous result. The LHS of (12) can be expanded in plane waves. Since we consider scattering at the negative light front, we must substitute $d^3x \rightarrow dx_+ d^2x_\perp$ [13] and accordingly $d^3p \rightarrow dp_- d^2p_\perp$. The expansion coefficient corresponds to the S matrix in momentum space, which is easily found to be

$$\begin{aligned} S(p', p) = & 2\pi\delta(p'_- - p_-) \left[\left(\frac{4}{(\vec{p}'_\perp - \vec{p}_\perp)^2} \right)^{1-iZ\alpha} \Gamma^2(1-iZ\alpha) \sin(\pi iZ\alpha) \right. \\ & \left. + (2\pi)^2 \delta(\vec{p}'_\perp - \vec{p}_\perp) \right] \bar{u}(p') \hat{\gamma}_- u(p) \end{aligned} \quad (16)$$

Here u denotes the electron unit spinor. If the trajectory of the ion is shifted by the impact parameter \vec{b} , this result is simply multiplied by the factor $e^{i(\vec{p}'_\perp - \vec{p}_\perp) \cdot \vec{b}}$. p and p' are the incoming and outgoing momenta. We note that the negative light cone momentum $p_- = p_0 - p_3$ is conserved in the scattering. The positive light cone momentum is fixed by the mass shell condition. The first term in the square brackets in (16) corresponds to the T matrix. Eq. (16) represents a well known result which was previously derived in e.g. [1, 4, 3, 7].

2.2 Perturbative Approach

In this section we want to derive the eikonal form of the T matrix via perturbation theory. Several approximations are necessary to obtain the eikonal form, namely the neglect of the

¹The effect of the potential (10) also can be described within the Aichelburg-Sexl metric. Two field-free regions of space-time meet at $z = t$, such that (the superscripts $<$ and $>$ denote $t > z$ and $t < z$, respectively)

$$x_\perp^> = x_\perp^< \quad (13)$$

$$z^> = z^< - Z\alpha \ln x_\perp^2 \quad (14)$$

$$t^> = t^< - Z\alpha \ln x_\perp^2 \quad (15)$$

The result (12) is then easily obtained by simply substituting (13)-(15) into the plane wave at $t > z$.

longitudinal components of the photon momentum, the conservation of the photon light cone momentum, as well as the simplification of the matrix structure of the interaction [5]. The calculation shows that these approximations are the counterparts of the requirement of a vanishing longitudinal vector potential and the step function dependence of the transverse vector potential. Having this in mind we directly use the asymptotic high-energy expression of the potential. We then evaluate the terms of the perturbation series for the external-field scattering problem depicted by the Feynman graphs of Fig. 1.

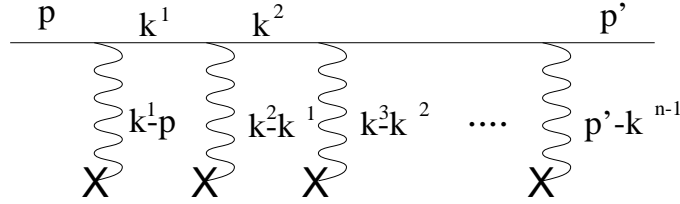


Figure 1: Scattering of an electron at an external potential.

The potential entering into our calculations is of the form

$$V_0(x) = V_3(x) = \delta(z - t)V_\perp(\vec{x}_\perp) \quad (17)$$

In the following calculations it will not be necessary to specify the explicit form of $V_\perp(\vec{x}_\perp)$. Problems related to the logarithmic potential obtained in the last section will be discussed in section 4. We use the light-cone variables

$$\begin{pmatrix} p_- \\ p_+ \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} p_0 \\ p_3 \end{pmatrix} \quad (18)$$

The Feynman propagator describing the internal electron lines reads

$$S_F(p) = \frac{1}{\hat{\gamma}_0 p_0 - \hat{\vec{\gamma}} \cdot \vec{p} - m + i\epsilon} = \frac{\frac{1}{2}(\hat{\gamma}_- p_+ + \hat{\gamma}_+ p_-) - \hat{\vec{\gamma}}_\perp \cdot \vec{p}_\perp + m}{p_+ p_- - p_\perp^2 - m^2 + i\epsilon} \quad (19)$$

The following products of gamma matrices in the light-cone representation are needed:

$$\begin{aligned} \hat{\gamma}_\pm \hat{\gamma}_\mp \hat{\gamma}_\pm &= 4\hat{\gamma}_\pm \\ \hat{\gamma}_\pm \hat{\vec{\gamma}}_\perp \hat{\gamma}_\mp &= 2\hat{\vec{\alpha}}_\perp \hat{\gamma}_\mp \\ \hat{\gamma}_\pm \hat{\vec{\gamma}}_\perp \hat{\gamma}_\pm &= 0 \\ \hat{\gamma}_\pm \hat{\gamma}_\mp &= 2\hat{\gamma}_0 \hat{\gamma}_\mp \\ \hat{\gamma}_\pm \hat{\gamma}_\pm &= 0 \end{aligned} \quad (20)$$

The amplitude for electron scattering in first order perturbation theory is

$$A_{p'p}^{(1)} = (2\pi)(-i)\delta(p'_- - p_-)F_{p'p}(V_\perp)\bar{u}(p')\hat{\gamma}_- u(p) \quad (21)$$

$F_{p'p}(\)$ denotes the Fourier transform with respect to the transverse coordinates taken at the momentum $(\vec{p}'_\perp - \vec{p}_\perp)$

$$F_{p'p}(V_\perp) = \int d^2 x_\perp e^{-i\vec{x}_\perp \cdot (\vec{p}'_\perp - \vec{p}_\perp)} V_\perp(\vec{x}_\perp) \quad (22)$$

In second order the amplitude reads

$$\begin{aligned}
A_{p'p}^{(2)} &= \int \frac{dk_+ dk_- d^2 k_\perp}{(2\pi)^4} (2\pi)^2 (-i)^2 i \delta(k_- - p_-) \delta(p'_- - k_-) \frac{k_-}{k_- k_+ - k_\perp^2 - m^2 + i\epsilon} \\
&\quad F_{kp}(V_\perp) F_{p'k}(V_\perp) \bar{u}(p') \hat{\gamma}_- u(p) \\
&= (2\pi) (-i)^2 \delta(p'_- - p_-) \frac{1}{2} F_{p'p}(V_\perp^2) \bar{u}(p') \hat{\gamma}_- u(p)
\end{aligned} \tag{23}$$

The k_+ integral in (23) drops out using the symbolic substitution

$$1/(x + i\epsilon) \rightarrow P(1/x) - i\pi\delta(x) \tag{24}$$

since the principal value integral P vanishes. It is interesting to note that the simple structure of the results (21) and (23) is retained if one goes to higher orders of perturbation theory. The n^{th} order amplitude factorizes into $n - 1$ integrals of the form (23) which leads to

$$A_{p'p}^{(n)} = (2\pi) (-i)^n \delta(p'_- - p_-) \frac{1}{n!} F_{p'p}(V_\perp^n) \bar{u}(p') \hat{\gamma}_- u(p) \tag{25}$$

This result is obtained by symmetrizing the $n - 1$ integrals over the positive light cone momenta in (24) yielding the expression $(-i2\pi)^{n-1}/n! \prod_i \delta(k_+^i)$ [5]. This corresponds to reconsidering the different time orderings and finally dividing by $n!$ to prevent double counting. This symmetrization procedure directly shows that the principal value terms in (24) do not contribute.

Clearly, with (25) the perturbation series can be summed up to yield the result

$$A_{p'p} = 2\pi \delta(p'_- - p_-) \mathcal{T}(\vec{p}'_\perp - \vec{p}_\perp) \bar{u}(p') \hat{\gamma}_- u(p) \tag{26}$$

Here we defined the momentum transfer function

$$\mathcal{T}(\vec{p}'_\perp - \vec{p}_\perp) = F_{p'p}(e^{-iV_\perp(\vec{x}_\perp)} - 1) \tag{27}$$

with

$$V_\perp(\vec{x}_\perp) = \int_{-\infty}^{+\infty} dt V_0(x) \tag{28}$$

This result reproduces the eikonal form.

3 Solution in the field of several ions

3.1 The case of two colliding ions

In the c.m. frame, the field of two ultrarelativistic colliding ions A and B , cf. Figure 2, reads

$$V_{0/3}(x) = \delta(z - t) V_\perp^A(\vec{x}_\perp) \pm \delta(z + t) V_\perp^B(\vec{x}_\perp) \tag{29}$$

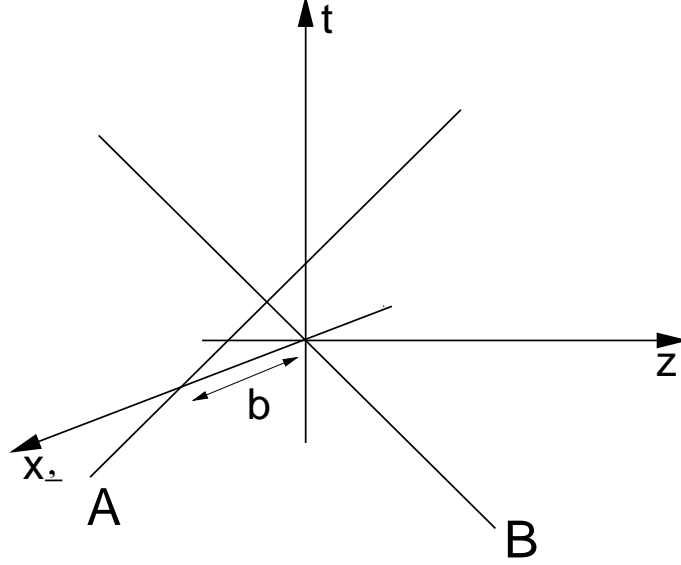


Figure 2: Geometry given by two lightlike ions colliding with an impact parameter \vec{b} . The x_\perp -axis symbolically denotes the transverse plane. The \vec{b} dependence of (29) is absorbed in the definitions of $V_\perp^{A,B}(\vec{x}_\perp)$.

The identity (4) can also be applied to potentials given by a superposition as is easily verified

$$(i\partial_x \mp i \sum \partial_x \ln \phi_i)^n = \left(\prod \phi_i\right)^{\pm 1} (i\partial_x)^n \left(\prod \phi_i\right)^{\mp 1} \quad (30)$$

Since in the case of (29) we have two discontinuities, the asymptotic solution is not obtained as easily as in section 2.1. The explicit calculation is shown in Appendix C.

It is found, that the two ions couple to distinct components of the electron spinor. We show in this section, how this behaviour follows from perturbation theory and how it can be interpreted consistently.

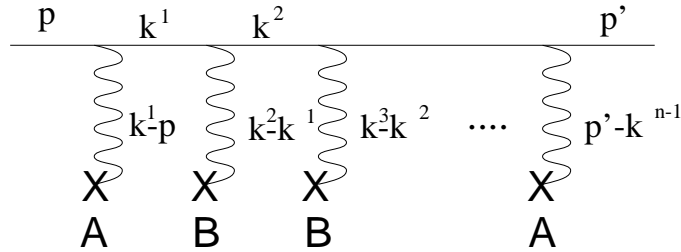


Figure 3: Same as Fig. 1 but considering two ions A and B as external sources. This diagram does not contribute in the high-energy limit.

We have to consider several new diagrams describing the alternate interaction of the electron with both ions. Using (20) we find, that the contribution to the T matrix of an arbitrary number of interactions with one ion that are sandwiched between interactions

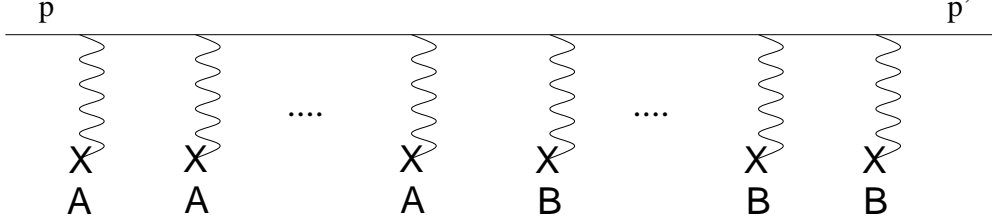


Figure 4: Class of Feynman graphs contributing in the high-energy limit.

with the other ion (see Fig. 3), vanishes. The reason is, that we end up with an integral of the form

$$A \sim \int \frac{dk_{\pm}}{(k_{\pm}p_{\mp} - k_{\perp}^2 - m^2 + i\epsilon)(k_{\pm}p'_{\mp} - k'_{\perp}^2 - m^2 + i\epsilon)} = 0 \quad (31)$$

The vanishing of this integral is immediately seen from Cauchy's theorem since the contour can be closed in the upper half plane, where the integrand is analytic.

In the ultrarelativistic limit the electron will therefore interact with the ions separately, see Figure 4. The separate interactions of the electron with the two ions A and B are linked in the following way

$$\begin{aligned} A_{p'p}^{tot} &= \int \frac{d^2k_{\perp}}{(2\pi)^2} \mathcal{T}_A(-\vec{p}_{\perp} + \vec{k}_{\perp}) \mathcal{T}_B(\vec{p}'_{\perp} - \vec{k}_{\perp}) \\ &\quad \bar{u}(p') \frac{-\hat{\alpha}_{\perp} \cdot \vec{k}_{\perp} + \gamma_0 m}{p'_+ p_- - k_{\perp}^2 - m^2 + i\epsilon} \hat{\gamma}_+ u(p) \\ &\quad + \int \frac{d^2k_{\perp}}{(2\pi)^2} \mathcal{T}_A(-\vec{p}_{\perp} + \vec{k}_{\perp}) \mathcal{T}_B(\vec{p}'_{\perp} - \vec{k}_{\perp}) \\ &\quad \bar{u}(p') \frac{-\hat{\alpha}_{\perp} \cdot (\vec{p}_{\perp} + \vec{p}'_{\perp} - \vec{k}_{\perp}) + \gamma_0 m}{p'_- p_+ - (\vec{p}_{\perp} + \vec{p}'_{\perp} - \vec{k}_{\perp})^2 - m^2 + i\epsilon} \hat{\gamma}_- u(p) \end{aligned} \quad (32)$$

Here we have already added both possible time orderings. \mathcal{T}_A and \mathcal{T}_B are the momentum transfer functions \mathcal{T} , defined in (26) for the interactions with ion A and B , respectively. This result is equivalently obtained by using the discontinuous behaviour at the light fronts (see Appendix C) and corresponds to the result of Segev and Wells [7].

To understand the decoupling property, one has to consider the matrix structure of the potential. To this end we write down the Dirac equation in the following form

$$\left[i\partial_t + \hat{\vec{\alpha}} \cdot i\vec{\nabla} - \hat{\gamma}_0 m - (1 \pm \beta \hat{\alpha}_z) A_0 \right] \psi = 0 \quad (33)$$

where the sign depends upon the direction of motion and A_0 is given by (1). In the limit $\beta \rightarrow 1$ the operators $1/2(1 \pm \beta \hat{\alpha}_z)$ become orthogonal projection operators [7]. The action of these operators can be understood if one recalls the standard form of Lorentz transformations [14] in spinor space

$$\psi'(x') = e^{-(i/4)\sigma_{\alpha\beta}\omega^{\alpha\beta}} \psi(x) \quad (34)$$

Here $\sigma_{\alpha\beta} = i/2[\gamma_\alpha, \gamma_\beta]$ and the exponent represents the product of the rapidity vector $\vec{\omega}$ times the generators of the Lorentz transformation. For a boost in $+z$ direction (34) simplifies to

$$\begin{aligned}\psi'(x') &= e^{-\frac{\omega}{2}\hat{\alpha}_z}\psi(x) \\ &= \cosh\left(\frac{\omega}{2}\right)\left(1 - \tanh\left(\frac{\omega}{2}\right)\hat{\alpha}_z\right)\psi(x)\end{aligned}\quad (35)$$

Therefore, (see (1)), a Lorentz-transformed vector acting in spinor space

$$(1 \pm \beta\hat{\alpha}_z)A_0 = \gamma(1 \pm \beta\hat{\alpha}_z)\frac{-Z\alpha}{r'}$$

can directly be obtained by a Lorentz transformation (34) accounting for the vectorial nature of the transformed object with a factor 2 in the exponent. The operators $(1 \pm \hat{\alpha}_z)$ are $1/\gamma$ times a Lorentz transformation with effectively infinite rapidity. These operators project the Dirac spinors onto causally disconnected subspaces of the Hilbert space. Therefore it is simply causally impossible for the Dirac spinor to communicate alternately with both ions.

Therefore, even the exact expressions for the interaction of an electron with two colliding ultrarelativistic ions maintains the structure of the two-photon graph. We can interpret (32) as the interaction of an electron in lowest order with a "dressed" potential of the form [15]

$$\tilde{V}_0(x) = \tilde{V}_3(x) \sim \delta(z - t) \left(\left(\frac{1}{x_\perp} \right)^{2iZ\alpha} - 1 \right) \quad (36)$$

An inspection of (32) reveals that the scattering amplitude is represented by a divergent integral. There are infrared divergencies caused by the poles of the momentum transfer functions \mathcal{T}_A and \mathcal{T}_B , see Appendix D. It is interesting to note, that an explicit introduction of a photon mass describing a screened Coulomb potential does *not* yield a regularized expression for the functions \mathcal{T} . On the other hand, if the modified potential (36) is screened with a damping factor $e^{-\epsilon x_\perp}$, this leads in momentum space to

$$\tilde{V}(k) \sim \frac{1}{(\epsilon^2 + k_\perp^2)^{(1-i\alpha Z)}} \Gamma(2(1 - i\alpha Z)) P_{1-2i\alpha Z}(\epsilon(\epsilon^2 + k_\perp^2)^{-\frac{1}{2}}) \quad (37)$$

which resembles the propagator of a photon with mass ϵ . $P_{1-2i\alpha Z}(\)$ denotes a Legendre function.

We should stress however, that such artificial regularization procedures are not needed. There is a natural cut off since the condition for the applicability of the used approximations requires (see Appendix B)

$$\gamma \gg \frac{x_\perp}{|z - \beta t|}$$

which in momentum space translates into the condition

$$k_\perp \gg \frac{\omega}{\gamma}$$

This lower bound for the transverse momentum corresponds to the cut off inherent in the Fourier transform of the potential, cf eq. (D.1) The introduction of this cut off has the important property to restore the energy dependence of the amplitude which was lost when taking the limit $\gamma \rightarrow \infty$.

3.2 Solution in the field of channeled ions

Here we want to sketch briefly an extension of the formalism discussed so far to the case of more than two colliding charges. The causal decoupling of interactions with sources moving on the positive and negative light-cone, respectively, and the above interpretation of the interaction can be used to calculate the scattering amplitude of electrons (or more realistically electron-positron pair production) for a field configuration which corresponds to the channeling of an ion in a crystal.

We use the equal speed system, the crystal is moving in $-z$ direction. The crystal layers have a spatial distance $a\vec{e}_z$. In the ultrarelativistic case, the electron again interacts with the ion and the crystal layers separately and we get simple time orderings of the interaction. For n crystal layers we have $n + 1$ possibilities. For the sake of simplicity we formulate the perturbative description of the successive interactions of the electron with both the ion and the crystal layers directly with modified potentials of the form (36). One then obtains for the interaction with two neighbouring crystal layers the integral

$$\begin{aligned}
A &= -\delta(p'_+ - p_+) e^{ip'_- \frac{a}{2}} \\
&\quad \int \frac{dk_- d^2 k_\perp}{(2\pi)^2} \frac{\bar{u}(p') \hat{\gamma}_+ u(p)}{-ik_- + i \frac{k_\perp^2 + m^2}{k_+} + \frac{\epsilon}{k_+}} e^{-ik_- \frac{a}{2}} \mathcal{T}_{C_i}(\vec{k}_\perp - \vec{p}_\perp) \mathcal{T}_{C_{i+1}}(\vec{p}'_\perp - \vec{k}_\perp) \\
&= 2\pi \delta(p'_+ - p_+) \int \frac{d^2 k_\perp}{(2\pi)^2} e^{i \left(-\frac{k_\perp^2 + m^2}{2p'_+} + \frac{p'_-}{2} \right) a} \\
&\quad \mathcal{T}_{C_i}(\vec{k}_\perp - \vec{p}_\perp) \mathcal{T}_{C_{i+1}}(\vec{p}'_\perp - \vec{k}_\perp) \bar{u}(p') \hat{\gamma}_+ u(p) \quad a > 0, \epsilon \rightarrow 0
\end{aligned} \tag{38}$$

The subscripts C_i and C_{i+1} denote the scattering amplitudes from the interaction of the electron with the i th and the $(i + 1)$ th crystal layer. For $a < 0$ (reverse direction of electron motion) the integral vanishes, which expresses, that the electron can not interact alternately with neighbouring crystal layers, due to causality. The derivation of this functional connection using (30) is shown in Appendix C.

If the electron interacts with the ion between interacting with two distinct crystal layers, we get

$$\begin{aligned}
A &= \int \frac{d^2 k_\perp d^2 k'_\perp}{(2\pi)^4} \frac{e^{ip'_- \frac{a}{2}} \left(e^{-i(k_\perp^2 + m^2 - i\epsilon) \frac{a}{2p_+}} - e^{-i(k'^2_\perp + m^2 - i\epsilon) \frac{a}{2p'_+}} \right)}{(p'_+(k_\perp^2 + m^2) - p_+(k'^2_\perp + m^2) - i\epsilon(p'_+ - p_+))} \\
&\quad \mathcal{T}_{C_i}(\vec{k}_\perp - \vec{p}_\perp) \mathcal{T}_A(\vec{k}'_\perp - \vec{k}_\perp) \mathcal{T}_{C_{i+1}}(\vec{p}'_\perp - \vec{k}'_\perp) \\
&\quad \bar{u}(p') (-\hat{\alpha}_\perp \cdot \vec{k}_\perp + \hat{\gamma}_0 m) (-\hat{\alpha}_\perp \cdot \vec{k}'_\perp + \hat{\gamma}_0 m) \hat{\gamma}_+ u(p)
\end{aligned} \tag{39}$$

Successive interactions with different crystal layers factorize and any scattering process including intermediate interaction with the channeled ion gives the same amplitude. Further studies will have to show how these considerations can be put to use for the calculation of pair creation in channeling.

4 Discussion

In the previous sections the potential of a fast moving charge has been substituted by its asymptotic high-energy expression. From a mathematical (and also a physical) point of view this is a problematic limit, since the required transformation is not an element of the Lorentz group. Furthermore, the potential (1), a bounded operator in Hilbert space, gets transformed into an unbounded operator, and finally the number of spatial dimensions gets reduced from three to two.

The ansatz directly reflects the approximations made by Chang and Ma [5] who neglected the longitudinal components of the photon momentum, giving the δ -functions for the respective conserved light cone momenta. The above mentioned problems emerge here in the fact that the longitudinal components of the photon momentum never really vanish.

All approximations allow the well known conclusion, that the eikonal expression can be regarded as the contribution of all ladder diagrams in the high-energy limit and that it is completely compatible with a perturbative calculation.

In the case of two ions we mainly profited from the causal decoupling of the interactions implied by the presence of the factors $(1 \pm \alpha_3)$. The matrix structure of the true interaction is given by $(1 \pm \beta \hat{\alpha}_3) \approx (1 \pm \hat{\alpha}_3) \mp \hat{\alpha}_3/2\gamma^2$, so that the leading corrections to this behaviour are suppressed with $1/\gamma^2$.

However, the considered calculations have inherent dangers. Fortunately we had to specify neither the transverse part of the potential nor its Fourier transform throughout our perturbative calculation. The first point may serve to generalize the validity of the result to any function $V_\perp(\vec{x}_\perp)$. However, a naive comparison even with the first order Born approximation would have failed due to the difficulties concerning the Fourier transform of the logarithm, whereas the logarithm in the argument of the exponential function is meaningful and correct. On the one hand the integration of the potential eliminates one dimension, which is finally recovered in the overall δ function for the light cone momenta. On the other hand the calculation of the scattering matrix between asymptotic states ($t \rightarrow \pm\infty$) corresponds to the (unphysical) limit $\gamma \rightarrow \infty$. The detour via the Fourier transform of the ungauged potential gives a logarithm as well (see Appendix D), depending, however, strongly upon a regularization mass μ . This has its root in the fact, that $1/(k_\perp^2 + \mu^2)$ is not the correct two-dimensional photon propagator [16]. The divergent term $\lim_{\mu \rightarrow 0} \ln \mu^2$ is the term \mathcal{C} in section 2.1.

Now it is well known, that two-dimensional fields in the limit of vanishing mass are rather ill-defined objects, whereas the exponential of these fields is not. The Fourier transform of this exponential expression is elementary (see (16)). It can further be expanded

into a Taylor series. Although it is not justified to identify the different terms with the Fourier transforms of the powers of the logarithm, the first term corresponds to the high energy limit of the Fourier transform of the retarded potential, which is rather accidental.

Nevertheless, the correct Fourier transform of the logarithm in two dimensions is obtained by Taylor expansion of $\mathcal{T}(k)$, but the limit $Z\alpha \rightarrow 0$ has to be taken *after* having integrated the expression with a test function [16].

$$\int d^2x_\perp e^{-i\vec{k}_\perp \cdot \vec{x}_\perp} \ln x_\perp^2 = \lim_{iZ\alpha \rightarrow 0} \frac{d}{d(iZ\alpha)} \left(\pi \frac{\Gamma(1 - i\alpha Z)}{\Gamma(i\alpha Z)} \left(\frac{4}{k_\perp^2} \right)^{1 - i\alpha Z} \right) \quad (40)$$

This peculiarity is related to the fact, that the linearity of the Fourier transform is not strictly defined for the action on infinite series, resulting in the non-commutability of limiting procedures as in (40). Another way is to rewrite the logarithm by the t integral of the gauged potential and expressing the integrand by means of its Fourier transform. One then finds for the Fourier transform of the logarithm in two dimensions [17]:

$$- \int d^2x_\perp e^{-i\vec{k}_\perp \cdot \vec{x}_\perp} \ln x_\perp^2 = \lim_{\lambda \rightarrow 0} 4\pi \left(\frac{1}{k_\perp^2 + \lambda^2} + \pi \delta^2(k_\perp) \ln \left(\frac{\lambda^2}{\mu^2} \right) \right) \quad (41)$$

with $\lambda = \omega/\gamma$, $\mu = 2/e^C$. The condition $\lambda \rightarrow 0$ coincides with the limit $\gamma \rightarrow \infty$ (see Appendix D). The correct treatment of this result again requires the limit to be taken after integrating the expression with a test function. Inverse transformation shows the required independence of the result of the regularization parameter λ .

In view of the previous discussion one may infer that it is not justified to identify the eikonal expression using a gauged potential with first-order perturbation theory using the original potential, since this ignores the gauge transformation applied to the potential.

In any case the above considerations show, that the limit $\gamma \rightarrow \infty$ is pathological and that its implications should be studied with care.

5 Conclusion

The transformation presented in section 2.1 directly yields the scattering amplitude for (arbitrary) particles scattered at fast charge centers. Due to Lorentz invariance the expression for the amplitude holds even for the case of static scattering centers, which gives the classical form of the eikonal approximation. The essential ingredients in the static case – the vanishing of the spin current and the assumption that ϕ was a slowly varying function – have been replaced by the discontinuous behaviour at the light fronts in the presented case of fast scattering centers.

The gauge transformed high energy limit of the potential directly contains the necessary approximations mentioned in section 2.2, and perturbative calculations using this potential can be done without further assumptions.

The results obtained by eikonal approximations or by application of the transformation (30) are shown to be equivalent to the sum of all ladder graphs. It should therefore not

be surprising that it is possible to regain perturbative results from the eikonal expression. We showed however, that the obtained results must be studied with care, and that a wrong treatment only accidentally leads to correct results.

Finally, the interaction of the electron with several ions moving along the light cones was shown to decouple due to causality.

The high-energy scattering amplitude of electrons in the field of two colliding ions has the same structure as the second-order perturbative result [8]. It can therefore be considered as a two-photon process with a modified potential of the form (36). The restrictions for the exchanged momenta imposed by the considered approximations are compatible with those encountered in the Weizsäcker–Williams method of virtual quanta. This method corresponds to a first-order Born approximation in the temporal gauge, considering only the transverse part of the interaction (the longitudinal part is suppressed by $1/\gamma^2$). The analogy of the obtained scattering amplitude to the two-photon process allows to adopt the approximations made in [8], aiming to express the result in the Weizsäcker–Williams form. The cross section of the scattering process can therefore be obtained from the Klein–Nishina formula for Compton-scattering and photon distributions obtained from (36) in the temporal gauge [18]. The simplified structure of the scattering amplitude allows for a study of the high-energy behaviour of electron-positron pair production, that accounts correctly for the Coulomb effects of both ions.

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A Transformation of the Dirac equation

According to (4) we set

$$A_0 = i\partial_t \ln \phi \tag{A.1}$$

$$A_3 = i\partial_z \ln \phi' \tag{A.2}$$

Since $A_3 = \beta A_0$ and $\partial_t = -\beta\partial_z$ due to the fact, that the t and z dependence enters only via the combination $(z - \beta t)$, we find

$$\frac{\partial_z \phi'}{\phi'} = -\beta^2 \frac{\partial_z \phi}{\phi} \tag{A.3}$$

$$\Rightarrow \phi' = \phi^{-\beta^2} \tag{A.4}$$

i.e. $\phi = 1/\phi'$ for $\beta \rightarrow 1$ as expected (see (4),(2)). Inserting this into the Dirac equation, we find

$$\left[\phi \hat{\gamma}_0 i \partial_t \frac{1}{\phi} + \phi \phi^{-\frac{1}{\gamma^2}} \hat{\gamma}_3 i \partial_z \frac{1}{\phi \phi^{-\frac{1}{\gamma^2}}} + \hat{\gamma}_\perp i \vec{\nabla}_\perp - m \right] \psi \quad (\text{A.5})$$

$$= \left[\phi \left(\hat{\gamma}_0 i \partial_t + \phi^{-\frac{1}{\gamma^2}} \hat{\gamma}_3 i \partial_z \frac{1}{\phi^{-\frac{1}{\gamma^2}}} \right) \frac{1}{\phi} + \hat{\gamma}_\perp i \vec{\nabla}_\perp - m \right] \psi \quad (\text{A.6})$$

$$= \phi \left[\hat{\gamma}_0 i \partial_t + \hat{\gamma}_3 \left(i \partial_z - \frac{1}{\beta^2 \gamma^2} A_3 \right) + \hat{\gamma}_\perp (i \vec{\nabla}_\perp + i \text{grad}_\perp \ln \phi) - m \right] \tilde{\psi} \quad (\text{A.7})$$

$$= 0 \quad (\text{A.8})$$

In the last step we introduced $\tilde{\psi} = \psi/\phi$.

In the ultrarelativistic limit terms of the order $1/\gamma^2$ are neglected. We end up with a Dirac equation coupled to a purely transverse vector potential exhibiting a Heaviside step function dependence $\sim \theta(t - z)$. These properties are essential for our considerations.

B Ultrarelativistic limit of the potential

In this section we want to discuss the limit $\beta \rightarrow 1$ of the potential (1). From section (2.1) we expect the asymptotic form of the potential to be

$$\lim_{\beta \rightarrow 1} \frac{-1}{\sqrt{(z-t)^2 + \frac{x_\perp^2}{\gamma^2}}} = \delta(z-t) \ln(x_\perp^2) + \mathcal{C}' \quad (\text{B.1})$$

t integration of (1) determines \mathcal{C}'

$$\mathcal{C}' = \frac{-1}{|z-t|} - \delta(z-t) \ln(\gamma^2) \quad (\text{B.2})$$

where we had to require $\gamma \gg x_\perp/|z - \beta t|$.

An attempt to derive the limit by means of Fourier transformation of (1) with respect to z was presented in [3]. The Fourier transform reads

$$\begin{aligned} \int dz e^{i\omega z} \frac{1}{\sqrt{(z - \beta t)^2 + \frac{x_\perp^2}{\gamma^2}}} &= e^{i\omega \beta t} \int dz e^{i\omega z} \frac{1}{\sqrt{z^2 + \frac{x_\perp^2}{\gamma^2}}} \\ &= 2e^{i\omega \beta t} K_0 \left(\frac{\omega x_\perp}{\gamma} \right) \\ &\xrightarrow{\gamma \rightarrow \infty} -2e^{i\omega t} \ln \left(\frac{\omega x_\perp e^C}{2\gamma} \right) \end{aligned} \quad (\text{B.3})$$

The quantity C here denotes Euler's constant. The inverse Fourier transformation of this expression yields

$$\lim_{\beta \rightarrow 1} \frac{1}{\sqrt{(z-t)^2 + \frac{x_\perp^2}{\gamma^2}}} = \frac{1}{|z-t|} + g(x_\perp) \delta(z-t) \quad (\text{B.4})$$

The coefficient $g(x_\perp)$ of the delta distribution in this result is not uniquely specified. Naive application of the textbook formula [19]

$$\int \frac{dk}{2\pi} \ln |k| e^{ikx} = -\frac{1}{2|x|} \quad (\text{B.5})$$

would give $g(x_\perp) = 2[\ln(x_\perp/2\gamma) + C]$, but (B.5) is valid only up to arbitrary multiples of $\delta(x)$. The validity of (B.3), however, as well demands the condition $\gamma \gg \omega x_\perp$.

It is possible to find a gauge transformation that removes both, the long-range potential $1/|z-t|$ as well as $\delta(z-t) \ln(\gamma^2)$ (B.2). This is fulfilled with the gauge transformation [11]

$$\psi' = e^{iZ\alpha \ln(\gamma(z-t) + \sqrt{1+\gamma^2(z-t)^2})} \psi \quad (\text{B.6})$$

The gauge-transformed potential reads

$$A'_0 = -\frac{Z\alpha\gamma}{\sqrt{\gamma^2(z-\beta t)^2 + x_\perp^2}} + \frac{Z\alpha\gamma}{\sqrt{\gamma^2(z-\beta t)^2 + 1}} \quad (\text{B.7})$$

and has the ultrarelativistic limit

$$\lim_{\beta \rightarrow 1} A'_0 = +\delta(z-t) \ln(x_\perp^2) \quad (\text{B.8})$$

The appearance of the logarithm follows immediately from the inhomogeneous Maxwell equations in the Lorentz gauge that reduces to a two dimensional Poisson equation in the limit $\beta \rightarrow 1$.

This gauge transformation has the advantage to yield a short-range potential that allows for asymptotic plane wave solutions. For this reason it was used to obtain a faster convergence in coupled channel calculations [20].

C Solution of the Dirac equation with two discontinuities

The Dirac equation for an electron moving in the field of two ultrarelativistic colliding ions A and B reads

$$\left[\frac{1}{2} (\hat{\gamma}_- i\partial_{\tau_-} + \hat{\gamma}_+ i\partial_{\tau_+}) + i\hat{\gamma}_\perp \cdot \vec{\nabla}_\perp - m - \frac{1}{2} \hat{\gamma}_- \delta(\tau_-) V_\perp^A - \frac{1}{2} \hat{\gamma}_+ \delta(\tau_+) V_\perp^B \right] \psi = 0 \quad (\text{C.1})$$

where we used light cone variables $\tau_{\pm} = (t \pm z)/2$. One directly finds that $\hat{\gamma}_- \psi$ is discontinuous at τ_- through the action of ion A and $\hat{\gamma}_+ \psi$ is discontinuous at $\tau_+ = 0$ through the action of ion B , respectively.

We introduce $\psi_{\pm} = (1 \pm \hat{\alpha}_z) \psi$ and use $2\psi = \psi_- + \psi_+$ to formulate the problem as follows

$$\left(i\partial_{\tau_+} + i\hat{\vec{\alpha}}_{\perp} \cdot \vec{\nabla}_{\perp} - \gamma_0 m - \delta(\tau_+) V_{\perp}^B \right) \psi_+ + \left(i\partial_{\tau_-} + i\hat{\vec{\alpha}}_{\perp} \cdot \vec{\nabla}_{\perp} - \gamma_0 m - \delta(\tau_-) V_{\perp}^A \right) \psi_- = 0 \quad (\text{C.2})$$

where (C.1) has been multiplied by $2\hat{\gamma}_0$. This we rewrite as

$$\begin{aligned} & \left(i\partial_{\tau_+} - \delta(\tau_+) V_{\perp}^B \right) \psi_+ + \left(i\hat{\vec{\alpha}}_{\perp} \cdot \vec{\nabla}_{\perp} - \gamma_0 m \right) \psi_- \\ &= - \left(i\partial_{\tau_-} - \delta(\tau_-) V_{\perp}^A \right) \psi_- - \left(i\hat{\vec{\alpha}}_{\perp} \cdot \vec{\nabla}_{\perp} - \gamma_0 m \right) \psi_+ \end{aligned} \quad (\text{C.3})$$

By using the standard representation of Dirac matrices and simply rearranging the four equations (C.3) one obtains

$$\begin{aligned} & \left[\begin{pmatrix} \mathbb{1} & 0 \\ 0 & 0 \end{pmatrix} \left(i\partial_{\tau_+} - \delta(\tau_+) V_{\perp}^B \right) + \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{1} \end{pmatrix} \left(i\partial_{\tau_-} - \delta(\tau_-) V_{\perp}^A \right) - m \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \right. \\ & \left. + i\partial_x \begin{pmatrix} 0 & -\sigma_y \\ \sigma_y & 0 \end{pmatrix} + i\partial_y \begin{pmatrix} 0 & -i\sigma_x \\ i\sigma_x & 0 \end{pmatrix} \right] \tilde{\psi} = 0 \end{aligned} \quad (\text{C.4})$$

where $\tilde{\psi} = \begin{pmatrix} \psi_1 + \psi_3 \\ \psi_2 - \psi_4 \\ \psi_1 - \psi_3 \\ \psi_2 + \psi_4 \end{pmatrix}$

corresponding to an isomorphic linear transformation [7] with the matrix

$$\Lambda = \begin{pmatrix} \mathbb{1} & \sigma_z \\ \mathbb{1} & -\sigma_z \end{pmatrix} \quad (\text{C.5})$$

Since Λ is a bijection, each side of (C.3) has to be zero. Off the light fronts we therefore have the two equations

$$i\partial_{\tau_+} \psi_+ = (i\hat{\vec{\alpha}}_{\perp} \cdot \vec{\nabla}_{\perp} - \gamma_0 m) \psi_- \quad (\text{C.6})$$

$$i\partial_{\tau_-} \psi_- = (i\hat{\vec{\alpha}}_{\perp} \cdot \vec{\nabla}_{\perp} - \gamma_0 m) \psi_+ \quad (\text{C.7})$$

According to (12) the discontinuities at the light fronts are described by

$$\psi_-(\tau_- = 0^+) = \phi^A(x_{\perp}) \psi_-(\tau_- = 0^-) , \quad \psi_-(\tau_+ = 0^+) = \phi^B(x_{\perp}) \psi_+(\tau_- = 0^-)$$

ϕ^A and ϕ^B are defined by (6) using the scalar parts of the potentials of the ions A and B . Let us study the spinor ψ_+ , evaluated at the surface $\tau_+ = 0^+$:

$$\psi_+(\tau_+ = 0^+) = \phi^B \frac{i\hat{\vec{\alpha}}_{\perp} \cdot \vec{\nabla}_{\perp} - \gamma_0 m}{p_-} \psi_-(\tau_+ = 0^-) \quad (\text{C.8})$$

In the region $\tau_- > 0$ the electron already has interacted with ion A and we can write

$$\psi_+(\tau_+ = 0^+, \tau_- > 0) = \phi^B \frac{i\hat{\vec{\alpha}}_\perp \cdot \vec{\nabla}_\perp - \gamma_0 m}{p_-} \phi^A (1 - \alpha_z) \psi_p \quad (\text{C.9})$$

where ψ_p is the incoming plane wave at momentum p . This relation also can be obtained immediately from (30) and (C.6) for τ_+ and $\tau_- > 0$. The operator $i\partial_+$ in (C.6) has been replaced by its eigenvalue p_- , the incoming negative light cone momentum. This is possible since p_- is conserved in the interaction with ion A . The expansion of $\psi_+(\tau_+ = 0^+)$ in the plane-waves basis reads

$$\psi_+(\tau_+ = 0^+, \tau_- > 0) = \int \frac{dp'_+ d^2 p'_\perp}{(2\pi)^3} B(p', p) e^{-ip'_+ \tau_- + i\vec{p}'_\perp \cdot \vec{x}_\perp} u(p') \quad (\text{C.10})$$

where we substituted $d^3 p' \rightarrow dp'_+ d^2 p'_\perp$ [13]. According to (C.8) the expansion coefficients are

$$B(p', p) = \int_0^\infty d\tau_- \int d^2 x_\perp e^{ip'_+ \tau_- - i\vec{p}'_\perp \cdot \vec{x}_\perp} \phi^B \bar{u}(p') \frac{i\hat{\vec{\alpha}}_\perp \cdot \vec{\nabla}_\perp - \gamma_0 m}{p_-} \psi_-(\tau_+ = 0^-, \tau_- > 0) \quad (\text{C.11})$$

In the region $\tau_- > 0$, $\tau_+ < 0$ the wave function ψ_- is a freely propagating wave packet with a fixed light cone momentum p_- and a superposition of transverse momenta p_\perp . The mass shell condition requires $p_- p_+ = \vec{q}_\perp^2 + m^2$. In this way $\psi_-(\tau_+ < 0, \tau_- > 0)$ can be obtained from $\psi_-(\tau_+ < 0, \tau_- = 0^+)$. We have

$$\psi_-(\tau_- > 0) = \int \frac{d^2 q_\perp}{(2\pi)^2} e^{-i\frac{q_\perp^2 + m^2}{p_-} \tau_- - ip_- \tau_+ + i\vec{q}_\perp \cdot \vec{x}_\perp} \int d^2 x'_\perp e^{i\vec{x}'_\perp \cdot (\vec{p}_\perp - \vec{q}_\perp)} \phi^A (1 - \hat{\alpha}_z) u(p) \quad (\text{C.12})$$

which leads to

$$\begin{aligned} B(p', p) &= i \int \frac{d^2 q_\perp}{(2\pi)^2} \int d^2 x'_\perp e^{i\vec{x}'_\perp \cdot (\vec{q}_\perp - \vec{p}'_\perp)} \phi^B \int d^2 x_\perp e^{i\vec{x}_\perp \cdot (\vec{p}_\perp - \vec{q}_\perp)} \phi^A \\ &\quad \bar{u}(p') \frac{\hat{\vec{\alpha}}_\perp \vec{q}_\perp - \gamma_0 m}{p'_+ p_- - q_\perp^2 - m^2 + i\epsilon} (1 - \hat{\alpha}_z) u(p) \end{aligned} \quad (\text{C.13})$$

Note, that the lower bound of the τ_- integration is 0, since we inserted the expression of ψ_- for $\tau_- > 0$.

Together with the corresponding term for the reverse order of interactions with the two ions, (C.13) is the S matrix for an electron scattered at the light fronts, first derived by Segev and Wells [7] in an elegant way using the transformation (C.5).

If both ions A and B move on positive light cones separated by the spatial distance $a\vec{e}_z$ (see section 3.2), we obtain with (30) for the interacting part of the spinor ψ

$$\begin{aligned} \psi_+(\tau_+ - a/2 = 0^+) &= \phi^A \psi_+(\tau_+ - a/2 = 0^-) \\ &= \phi^A \int \frac{d^2 q_\perp}{(2\pi)^2} e^{-i\frac{q_\perp^2 + m^2}{p_+} \frac{a}{2} - ip_+ \tau_- + i\vec{q}_\perp \cdot \vec{x}_\perp} \\ &\quad \int d^2 x'_\perp e^{i\vec{x}'_\perp \cdot (\vec{p}_\perp - \vec{q}_\perp)} \phi^B (1 + \hat{\alpha}_z) u(p) \end{aligned} \quad (\text{C.14})$$

The expansion of ψ_+ in plane waves at the point $\tau_+ = a/2 + 0^+$ yields the S matrix of this process in momentum space

$$S(p', p) = 2\pi\delta(p'_+ - p_+)i \int \frac{d^2q_\perp}{(2\pi)^2} e^{i\left(-\frac{q_\perp^2 + m^2}{2p'_+} + \frac{p'_-}{2}\right)a} \int d^2x_\perp e^{i\vec{x}_\perp \cdot (\vec{q}_\perp - \vec{p}'_\perp)} \phi^A \int d^2x_\perp e^{i\vec{x}_\perp \cdot (\vec{p}_\perp - \vec{q}_\perp)} \phi^B \bar{u}(p')(1 + \hat{\alpha}_z)u(p) \quad (\text{C.15})$$

in accordance with section 3.2.

D The photon propagator at high collision energies

The four-dimensional Fourier transform of the potential (1) reads

$$\int d^4x e^{ikx} \frac{-Z\alpha\gamma}{\sqrt{\gamma^2(z - \beta t)^2 + \vec{x}_\perp^2}} = -(2\pi)^2 Z\alpha\delta(k_0 - \beta k_3) \frac{2}{\left(\frac{k_3}{\gamma}\right)^2 + k_\perp^2} \quad (\text{D.1})$$

which has the following low and high-velocity limits

$$\lim_{\beta \rightarrow 0} = -(2\pi)^2 Z\alpha\delta(k_0) \frac{2}{|\vec{k}|^2} \quad (\text{D.2})$$

$$\lim_{\beta \rightarrow 1} = -(2\pi)^2 Z\alpha\delta(k_0 - k_3) \frac{2}{k_\perp^2} \quad (\text{D.3})$$

The last expression reflects the observation, that in the high-energy limit the longitudinal components k_- and k_+ of the photon momentum can be dropped.

After having performed the gauge transformation (B.6) and taken the limit $\gamma \rightarrow \infty$, the potential to be transformed is expression (B.8). Grignani and Mintchev [16] have shown, that it is wrong to identify the Fourier transform of (B.8) with (D.3) or with the regulated expression $1/(k_\perp^2 + \mu^2)$ with a regulating mass inserted by hand.

Calculating the time integral of A_0 in the eikonal expression and using (D.1) one finds

$$\int_{-\infty}^{\infty} dt A_0 = Z\alpha \ln(x_\perp^2) + Z\alpha \lim_{\mu \rightarrow 0} \ln \mu^2 \quad (\text{D.4})$$

with $1/2e^C$ absorbed in μ as in [3]. The term $\lim_{\mu \rightarrow 0} Z\alpha \ln \mu^2$ is the term \mathcal{C} in section 2.1 and is completely different from μ in eq (7) of [3]!

One may attempt to calculate the two-dimensional Fourier transform of the logarithm from a Taylor expansion in powers of $iZ\alpha$ of the Fourier transform of the T matrix (i.e. its transverse part) which is given by the following closed expression

$$\mathcal{T}(k_\perp) = \left(\frac{4}{k_\perp^2}\right)^{1-i\alpha Z} \Gamma^2(1-i\alpha Z) \sin(\pi i\alpha Z) \quad (\text{D.5})$$

$$= \pi \frac{\Gamma(1-i\alpha Z)}{\Gamma(i\alpha Z)} \left(\frac{4}{k_\perp^2}\right)^{1-i\alpha Z} \quad (\text{D.6})$$

The first terms of the Taylor expansion read

$$\begin{aligned}\mathcal{T}(k_\perp) \approx & +4\pi i\alpha Z \frac{1}{k_\perp^2} + 4\pi(i\alpha Z)^2 \frac{\ln(k_\perp^2/4) + C}{k_\perp^2} \\ & + 2\pi(i\alpha Z)^3 \frac{\ln^2(k_\perp^2/4) + 4C \ln(k_\perp^2/4) + 4C^2}{k_\perp^2} + \dots\end{aligned}\quad (\text{D.7})$$

The second term would then correspond to the desired Fourier transform (times $(iZ\alpha)$), the third term correspondingly to $(iZ\alpha)^2$ times the Fourier transform of the square of the transverse part of the potential (B.8) that has to be compared with the result of Torgerson [2].

This is, however, not justified, since the linearity of the Fourier transform is only guaranteed for finite sums and causes problems when applied to infinite series like the Taylor expansion of the exponential function. To get the correct result for the exact two dimensional euclidean photon propagator, the limit $iZ\alpha \rightarrow 0$ in

$$\int d^2x_\perp e^{-i\vec{k}_\perp \cdot \vec{x}_\perp} \ln x_\perp^2 = \lim_{iZ\alpha \rightarrow 0} \frac{d}{d(iZ\alpha)} \left(\pi \frac{\Gamma(1-i\alpha Z)}{\Gamma(i\alpha Z)} \left(\frac{4}{k_\perp^2} \right)^{1-i\alpha Z} \right) \quad (\text{D.8})$$

has to be taken after having integrated the result with a test function. Performing the limit without this precaution gives the wrong result (D.3).

Another form of the correct Fourier transform was derived in [17]. We obtain the equivalent form from the gauged potential A'_0 in (B.7). Since

$$-\ln x_\perp^2 = \lim_{\gamma \rightarrow \infty} \int_{-\epsilon}^{\epsilon} dt \left(\frac{\gamma}{\sqrt{\gamma^2 t^2 + x_\perp^2}} - \frac{\gamma}{\sqrt{\gamma^2 t^2 + 1}} \right) \quad (\text{D.9})$$

(ϵ is arbitrary but finite) and

$$\int dt d^2x_\perp e^{i\omega t - i\vec{k}_\perp \cdot \vec{x}_\perp} \left(\frac{\gamma}{\sqrt{\gamma^2 t^2 + x_\perp^2}} - \frac{\gamma}{\sqrt{\gamma^2 t^2 + 1}} \right) = 4\pi \left(\frac{1}{\left(\frac{\omega}{\gamma}\right)^2 + k_\perp^2} - 2\pi K_0\left(\frac{\omega}{\gamma}\right) \right) \quad (\text{D.10})$$

we find by direct substitution

$$-\int d^2x_\perp e^{-i\vec{k}_\perp \cdot \vec{x}_\perp} \ln x_\perp^2 = \lim_{\lambda \rightarrow 0} 4\pi \left(\frac{1}{k_\perp^2 + \lambda^2} + \pi \delta^2(k_\perp) \ln \left(\frac{\lambda^2}{\mu^2} \right) \right) \quad (\text{D.11})$$

with $\lambda = \omega/\gamma$, $\mu = 2/e^C$. The limit has to be treated in the same way as in (D.8).

When naively taking the limit $iZ\alpha \rightarrow 0$ immediately, by chance one obtains the high energy limit of the ungauged potential.

References

- [1] H.D.I. Abarbanel, C. Itzykson, Phys. Rev. Lett. 23 (1969) 53
- [2] R. Torgerson, Phys. Rev. 143 (1966) 1194
- [3] R. Jackiw, D. Kabat, M. Ortiz, Phys. Lett. B 277 (1992) 148
- [4] G. 't Hooft, Phys. Lett. B 198 (1987) 61
- [5] S.J. Chang, S.K. Ma, Phys. Rev. 188 (1969) 2385; S.J. Chang, S.K. Ma, Phys. Rev. Lett. 22 (1969) 1336
- [6] S.J. Chang, P.M. Fishbane, Phys. Rev. D 2 (1970) 1104
- [7] B. Segev, J.C. Wells, Phys. Rev. A 57 (1998) 1849
- [8] C. Bottcher, M.R. Strayer, Phys. Rev. D 39 (1989) 1330
- [9] D.C. Ionescu, J. Eichler, Phys. Rev. A 48 (1993) 1176
- [10] R.M. Miura, J. Math. Phys. 9 (1968) 1202
- [11] P.C. Aichelburg, R.U. Sexl, Gen. Rel. Grav. 2 (1971) 303
- [12] H.M. Fried, Functional Methods and Models in Quantum Field Theory, MIT Press (1972)
- [13] E. Meggiolaro, Phys. Rev. D 53 (1996) 3835
- [14] C. Itzykson, J.B. Zuber, Quantum Field Theory, McGraw-Hill (1985)
- [15] A.J. Baltz, Phys. Rev. Lett. 78 (1997) 1231
- [16] G. Grignani, M. Mintchev, Phys. Rev. D 38 (1988) 3163
- [17] R. Ferrari, Nuovo Cimento A 19 (1974) 204
- [18] U. Eichmann, J. Reinhardt, W. Greiner, manuscript in preparation
- [19] M.J. Lighthill, Introduction to Fourier analysis and generalized functions, Cambridge Univ. Press (1959)
- [20] N. Toshima, J. Eichler, Phys. Rev. A 42 (1990) 3896